Math 347 Worksheet

Worksheet 11: Permutations of n elements

November 7, 2018

1) Consider S_n the set of permutations of n elements.

   (i) Prove that any element $s \in S_n$ can be written as the composite

   $$s = t^1 \circ \dots t^k,$$

   where each t^i is a transposition.

   (ii) Fix $\ell \in [n]$. Prove that the transpositions in the above can all be taken to be

   $$t^i = t_{\ell,j}$$

   for some $j \in [n]$, where $t_{\ell,j}$ is the transposition that swaps ℓ and j.

2) Prove that for any element $s \in S_n$ there exists $k \leq n$ such that[1]

   $$s^k = e.$$

3) Draw a functional digraph of an element $s \in S_n$. What does the condition from the previous exercise mean in terms of the graph?

4) For a natural number n a *partition* of n is a way of writing n as a sum of positive numbers.

   (i) list the partitions of 6;

   (ii) prove that the number of partitions of n with k parts equals the number of partitions of n with largest part k.

5) (Extra) Consider a square in $\mathbb{R}^2$ with vertices $(1,1), (1,-1), (-1,-1)$ and $(-1,1)$. Let r be the function $r : \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation of 90 degrees, let $s : \mathbb{R}^2 \to \mathbb{R}^2$ be the function that reflects the points around the line $x = y$ (Make a drawing.). Let D_4 be the set of functions obtained by considering compositions of r or s any number of times.

   (a) prove that D_4 is finite and compute its size;

   (b) prove that any element $a \in D_4$ preserves the square;

   (c) prove that D_4 is not equal to S_n for any n. Can you find an n such that $D_4 \subset S_n$?

[1] Here s^k means one composes s with itself k times.